Numerical Calculation of the Potential Distribution in Ion Slit Lens Systems II

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The potential distribution is computed in certain ion slit lens systems, consisting of three parallel slits in three parallel electrodes. In a previous paper [1] the case was treated where the slit widths were smaller than the distances to the neighbouring electrodes. In the present paper this requirement has been dropped; for the sake of simplicity, however, the computations are confined to the case, where the central electrode represents a plane of symmetry.

Various approximation and iteration methods are given to find the necessary parameters to perform the SCHWARZ-CHRISTOFFEL transformation. Several typical examples are given.

In a previous paper [1] formulae were derived to compute the potential distribution in slit lens systems, consisting of three slits and satisfying the following conditions:

1. parallel electrodes; 2. infinite slit lengths; 3. electrode thicknesses very small, as compared with the electrode distances and slit widths; 4. all centres of the slits in one plane perpendicular to the electrodes.

The method was well suitable in the case, that the slit widths were smaller than the distances to the neighbouring electrodes.

In the present paper this requirement has been dropped and the potential distribution is computed in slit lens systems, where at least one slit is broader than the distance to a neighbouring electrode. For the sake of simplicity, however, the computations are confined to the case, where the central electrode represents a plane of symmetry. More general cases will be treated in a subsequent paper.

In the previous paper, the cardinal problem was the calculation of five quantities a, b, c, d, and e, governing the conformal mapping of the slit system. These quantities were found as the solution of the five equations:

$$\left.\begin{aligned}
a+r_1\ln\frac{a+b}{a-b}+r_2\ln\frac{a+d}{a-d}+\frac{a\,c^2\,e^2}{b^2\,d^2}&=s_1\,,\\
\frac{a^2-b^2}{2\,b}\,\frac{b^2-c^2}{b^2-d^2}\,\frac{b^2-e^2}{b^2}&=r_1\,,\\
c+r_1\ln\frac{b+c}{b-c}+r_2\ln\frac{c+d}{c-d}+\frac{a^2\,c\,e^2}{b^2\,d^2}&=s_2\,,\\
\frac{c^2-d^2}{2\,d}\,\frac{a^2-d^2}{b^2-d^2}\,\frac{d^2-e^2}{d^2}&=r_2\,,\\
e+r_1\ln\frac{b+e}{b-e}+r_2\ln\frac{d+e}{d-e}+\frac{a^2\,c^2\,e}{b^2\,d^2}&=s_3\,,
\end{aligned}\right\}\qquad(1)$$

where $2\,s_1$, $2\,s_2$ and $2\,s_3$ were the successive slit widths and $\pi\,r_1$ and $\pi\,r_2$ the electrode distances.

From the quantities a, b, c, d, and e the potential distribution and field strength were found in the following way:

The complex parameter $w=u+i\,v$ was introduced, connected to the coordinates x and y in the slit system through the formula:

$$z=x+i\,y=w+r_1\ln\frac{w+b}{w-b}+r_2\ln\frac{w+d}{w-d}+\frac{a^2\,c^2\,e^2}{b^2\,d^2}\,\frac{1}{w}\,.\qquad(2)$$

The potential distribution was computed as a linear superposition of the contributions of all electrodes. The contribution of one half electrode was given by

$$V(u,v)=\frac{V_n}{\pi}\left\{\operatorname{arc\,tan}\frac{v}{u-p_n}-\operatorname{arc\,tan}\frac{v}{u-q_n}\right\}\qquad(3)$$

with V_n = potential of the particular half electrode, p_n and q_n = the values of w, attributed to the infinite points at both sides of the half electrode (resp. $+\infty$, $+b$; $+b$, $+d$; $+d$, 0; 0, $-d$; $-d$, $-b$; $-b$, $-\infty$).

The field strength was found by differentiation.

We now have to solve the equations (1), without the restriction to the case of slit widths, smaller than the disctances to the neighbouring electrodes. We will confine ourselves to the symmetrical case, where $s_1=s_3$ and $r_1=r_2=r$. As is proved in the appendix, these restrictions imply that

$$a/b=d/e=w_1\,,\qquad b/c=c/d=w_2\,,\qquad(4)$$

[1] A. J. H. BOERBOOM, Z. Naturforschg. **14 a**, 809 [1959]; in this paper it is named I.

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and the equations (1) reduce to

$$\left.\begin{aligned} w_1^2 w_2^2+1+\frac{r}{e}\ln\frac{(w_1+1)(w_1 w_2^2+1)}{(w_1-1)(w_1 w_2^2-1)} &= \frac{s_1}{e}, \\ \frac{w_1^2-1}{2 w_1}\,\frac{w_1^2 w_2^4-1}{w_2^2+1} &= \frac{r}{e}, \\ 2 w_1 w_2+2\frac{r}{e}\ln\frac{w_2+1}{w_2-1} &= \frac{s_2}{e}. \end{aligned}\right\} \quad (5)$$

The formulae (2, 3) to compute the potential distribution and field strength remain essentially unchanged, substituting

$a=w_1^2 w_2^2 e;\quad b=w_1 w_2^2 e;\quad c=w_1 w_2 e;\quad d=w_1 e.$

In the previous paper it was shown, that the solution had to satisfy the condition that all unknowns were real and $a>b>c>d>e>0$, so $w_1>1$ and $w_2>1$. In the same paper the case corresponding to $w_1^2 \gg 1$ and $w_2^2 \gg 1$ was treated. If we drop these last requirements, there are three possibilities concerning the orders of magnitude of w_1^2 and w_2^2, viz.

a) $w_1^2 \gg 1$, but w_2^2 the same order of magnitude as unity, e. g. $1<w_2^2<10$,

b) $1<w_1^2<10$ and $w_2^2 \gg 1$,

c) $1<w_1^2<10$ and $1<w_2^2<10$.

In the following paragraphs, the solution of (5) will be given for these three cases.

1. Solution of the equations (5) for $w_1^2 \gg 1$

In the same way as in the first paper[1], to solve (5), we consider in first instance only the terms of highest order of magnitude. So in first approximation we get the equations:

$$\left.\begin{aligned} w_1^2 w_2^2+\frac{r}{e}\left(\frac{2}{w_1}+\frac{2}{w_1 w_2^2}\right) &= \frac{s_1}{e}, \\ \frac{w_1}{2}\,\frac{w_1^2 w_2^4}{w_2^2+1} &= \frac{r}{e}, \\ 2\frac{r}{e}\ln\frac{w_2+1}{w_2-1} &= \frac{s_2}{e}. \end{aligned}\right\} \quad (6)$$

Putting $s_1/4r=v_1$ and $s_2/4r=v_2$, the solution of this system is found to be

$$w_2=\coth v_2,\; w_1=\frac{w_2^2+1}{w_2^2}\,\frac{1}{v_1},\; e=\frac{s_1}{2 w_1^2 w_2^2}. \quad (7)$$

As no assumption has been made concerning the order of magnitude of w_2^2, these equations remain valid for $w_2^2 \gg 1$ in first approximation.

In Table 1 we give the function coth for the region in consideration, i. e. $1<w_2<3{,}5$. The function is plotted in Fig. 1.

If a more accurate solution is desired, a second approximation is necessary. In the previous paper we found series expansions in v_n^2, corresponding to $1/w_1^2$ and $1/w_2^2$. As in the present case only $1/w_1^2 \ll 1$, it seems appropriate to find series expansions in $1/w_1^2$ using w_2 as a parameter.

In this way we find:

$$\left.\begin{aligned} w_2 &= (\coth v_2)\left\{1+(1-\coth^{-4} v_2)\frac{1}{w_1^2}+\dots\right\}, \\ w_1 &= \frac{w_2^2+1}{w_2^2}\,\frac{1}{v_1}\left\{1+\frac{2 w_2^4+w_2^2+2}{3 w_2^4}\,\frac{1}{w_1^2}+\dots\right\}, \\ e &= \frac{s_1}{2 w_1^2 w_2^2}\left\{1+\frac{w_2^4-w_2^2+1}{3 w_2^4}\,\frac{1}{w_1^2}+\dots\right\}. \end{aligned}\right\} \quad (8)$$

The exact solution is found by an iteration method. The approximate equations (4) are made exact by adding in the right hand members the neglected quantities. Thus we get the exact equations:

$$\left.\begin{aligned} 2 w_1^2 w_2^2 &= \frac{s_1}{e}+\frac{1}{e}\left\{e\,\frac{w_1^2 w_2^4-w_1^2 w_2^2+w_1^2-1}{w_1^2 w_2^2}+r\left(\frac{2(w_2^2+1)}{w_1 w_2^2}-\ln\frac{(w_1+1)(w_1 w_2^2+1)}{(w_1-1)(w_1 w_2^2-1)}\right)\right\}=\frac{S_1}{e}, \\ \frac{w_1^3 w_2^4}{2(w_2^2+1)} &= \frac{r}{e}\left\{1+\frac{e}{r}\,\frac{w_1^2 w_2^4+w_1^2-1}{2 w_1(w_2^2+1)}\right\}=\frac{R}{e},\quad \ln\frac{w_2+1}{w_2-1}=\frac{s_2}{2r}-\frac{w_1 w_2}{r}e=\frac{S_2}{2R}. \end{aligned}\right\} \quad (9)$$

If we substitute in the right hand terms approximate values of w_1, w_2 and e a more accurate solution is found of a form analogous to (7):

$$w_2=\coth\frac{S_2}{4R},\; w_1=\frac{w_2^2+1}{w_2^2}\,\frac{4R}{S_1},\; e=\frac{S_1}{2 w_1^2 w_2^2}. \quad (10)$$

These new values of w_1, w_2 and e, substituted in the right hand terms of (9) furnish more accurate values for w_1, w_2 and e. In practice this iteration process converges quickly.

2. Solution of the equations (5) for $w_2^2 \gg 1$

In an analogous way as before we get the simplified equations:

$$\left.\begin{aligned} w_1^2 w_2^2+\frac{r}{e}\ln\frac{w_1+1}{w_1-1} &= \frac{s_1}{e}, \\ \frac{w_1^2-1}{2} w_1 w_2^2=\frac{r}{e},\quad 2 w_1^3 w_2 &= \frac{s_2}{e}. \end{aligned}\right\} \quad (11)$$

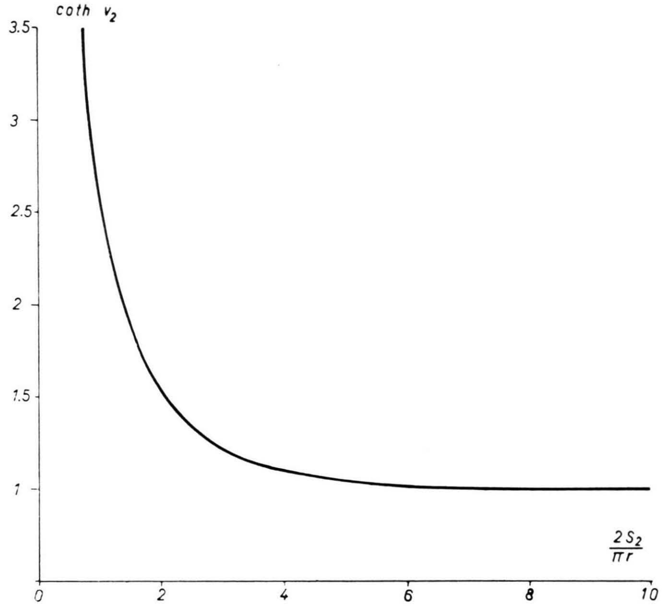

Fig. 1. The function coth v_2.

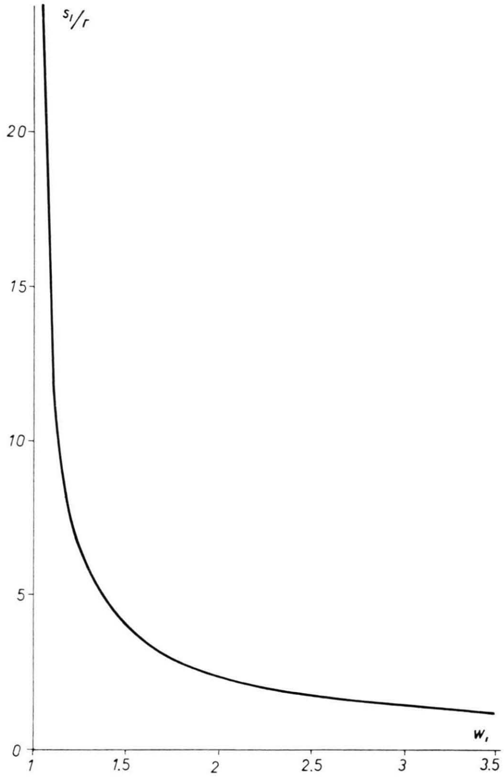

Fig. 2. The function $F(w_1)$ of formula (12).

$\frac{2\,s_2}{\pi r}$	$v_2 = \frac{s_2}{4r}$	coth v_2	$\frac{2\,s_2}{\pi r}$	$v_2 = \frac{s_2}{4r}$	coth v_2
10.00	3.9270	1.0008	2.50	0.98175	1.3266
7.50	2.9452	1.0056	2.00	0.78540	1.5249
5.00	1.9635	1.0402	1.50	0.58905	1.8897
4.00	1.5708	1.0903	1.25	0.49087	2.1983
3.50	1.3744	1.1368	1.00	0.39270	2.6764
3.00	1.1781	1.2094	0.75	0.29452	3.4941

Table 1.

w_1	$F(w_1)$	w_1	$F(w_1)$
1.01	105.801	1.30	5.8050
1.02	55.110	1.50	4.0094
1.03	38.041	2.00	2.4319
1.05	24.201	2.50	1.79970
1.10	13.5207	3.00	1.44315
1.20	7.8525	3.50	1.21001

Table 2.

From these equations we deduct:

$$\left.\begin{aligned} 2\,\frac{w_1}{w_1^2-1} + \ln\frac{w_1+1}{w_1-1} &= \frac{s_1}{r} = F(w_1), \\ w_2 = \frac{w_1^2}{w_1^2-1}\,\frac{4\,r}{s_2}, &\quad e = \frac{s_2}{2\,w_1^3\,w_2}. \end{aligned}\right\}(12)$$

As in I, the first function is tabulated in Table 2 and plotted in Fig. 2. From this table or graph w_1 is found and substituted in the next formula to give the value of w_2. From w_1 and w_2, by substitution in the third formula e is found.

Again these equations remain equally valid for $w_1^2 \gg 1$ in first approximation.

A more accurate solution is found by a series expansion, giving in second approximation:

$$\left.\begin{aligned} 2\,\frac{w_1}{w_1^2-1} + \ln\frac{w_1+1}{w_1-1} &= \frac{s_1}{r} - \frac{4\,w_1}{w_1^2-1}\,\frac{1}{w_2^2}, \\ w_2 &= \frac{w_1^2}{w_1^2-1}\,\frac{4\,r}{s_2}\left(1+\frac{w_1^2+2}{3\,w_1^2}\,\frac{1}{w_2^2}\right), \\ e &= \frac{s_2}{2\,w_1^3\,w_2}\left(1+\frac{2}{3}\,\frac{w_1^2-1}{w_1^2}\,\frac{1}{w_2^2}\right). \end{aligned}\right\}(13)$$

The exact solution is found by iteration of the equations:

$$\left.\begin{aligned}
&w_1^2 w_2^2 + \frac{w_1^2-1}{2} w_1 w_2^2 \ln\frac{w_1+1}{w_1-1} = \frac{s_1}{e}\left\{1 - \frac{e}{s_1} - \frac{r}{s_1}\ln\frac{w_1 w_2^2+1}{w_1 w_2^2-1} + \frac{e}{s_1}\,\frac{(w_1^2-1)(1+w_1^2 w_2^2)}{2\,w_1(w_2^2+1)}\ln\frac{w_1+1}{w_1-1}\right\} = \frac{S_1}{e},\\
&\qquad\qquad \frac{w_1^2-1}{2} w_1 w_2^2 = \frac{r}{e}\left\{1 + \frac{1}{w_2^2} + \frac{e}{r}\,\frac{w_1^2-1}{2\,w_1}\,\frac{1}{w_2^2}\right\} = \frac{R}{e},\\
&2\,w_1^3 w_2 = \frac{s_2}{e}\left\{1 + \frac{1}{w_2^2} + \frac{e}{s_2}(1-w_1^2 w_2^4)\frac{w_1^2-1}{w_1 w_2^2}\ln\frac{w_2+1}{w_2-1} + 2\frac{e}{s_2}\,\frac{w_1^3 w_2^2 - w_1 w_2^2 - w_1}{w_2}\right\} = \frac{S_2}{e}\\
&\quad = \frac{s_2}{e}\left\{1 + \frac{1}{w_2^2} + 2\frac{e}{s_2}\left[\frac{1}{3\,w_1 w_2}(-w_1^4 - 2\,w_1^2) + \frac{1}{w_1 w_2^3}(w_1^2-1) + \dots\right]\right\}.
\end{aligned}\right\}\quad(14)$$

3. General solving method for the equations (5)

If no assumption is made concerning the order of magnitude of w_1^2 or w_2^2, it seems rather laborous to give a direct solution of (5), though a series expansion is always possible. The following method gives the results more quickly.

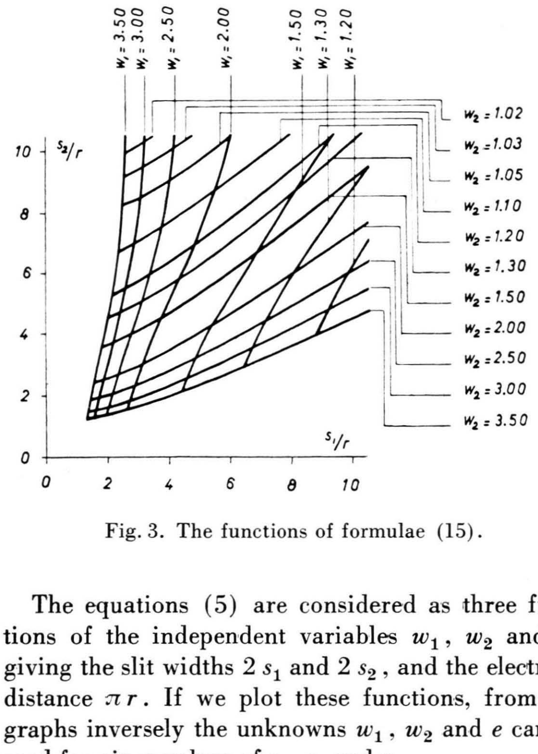

Fig. 3. The functions of formulae (15).

The equations (5) are considered as three functions of the independent variables w_1, w_2 and e, giving the slit widths $2\,s_1$ and $2\,s_2$, and the electrode distance πr. If we plot these functions, from the graphs inversely the unknowns w_1, w_2 and e can be read for given values of s_1, s_2 and r.

In the following graphs and tables the functions

$$\left.\begin{aligned}
\frac{s_1}{r} &= 2\,w_1\frac{(w_1^2 w_2^2+1)(w_2^2+1)}{(w_1^2 w_2^4-1)(w_1^2-1)}\\
&\qquad + \ln\frac{(w_1+1)(w_1 w_2^2+1)}{(w_1-1)(w_1 w_2^2-1)},\\
\frac{s_2}{r} &= \frac{4\,w_1^2 w_2(w_2^2+1)}{(w_1^2 w_2^4-1)(w_1^2-1)} + 2\ln\frac{w_2+1}{w_2-1}
\end{aligned}\right\}\quad(15)$$

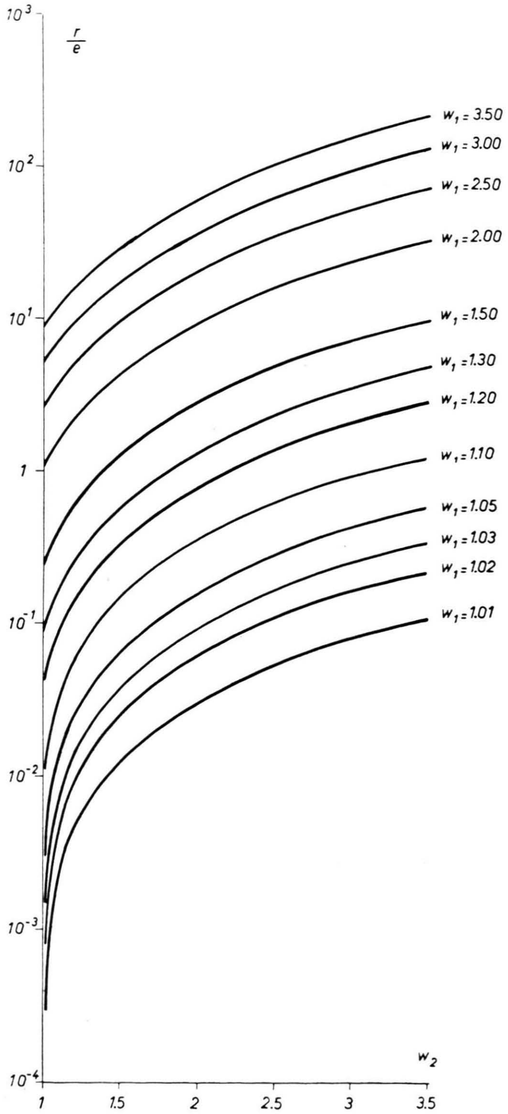

Fig. 4. The function r/e of formula (16).

are plotted and tabulated in the region $1 < w_1 \leqq 3{,}5$ and $1 < w_2 \leqq 3{,}5$.

w_2 \ w_1	1.01	1.02	1.03	1.05	1.10	1.20	1.30	1.50	2.00	2.50	3.00	3.50
1.01	6744	2553	1376.4	604.2	188.37	59.27	31.33	15.083	6.498	4.254	3.215	2.607
1.02	4066	1708.8	986.3	471.0	161.63	54.37	29.51	14.526	6.363	4.187	3.172	2.575
1.03	2920	1288.3	770.9	386.9	141.86	50.29	27.92	14.019	6.236	4.122	3.129	2.544
1.05	1879.2	868.9	539.9	286.9	114.59	43.92	25.28	13.134	6.004	4.003	3.050	2.484
1.10	1013.1	488.6	315.0	177.91	78.94	33.92	20.74	11.455	5.521	3.745	2.877	2.354
1.20	549.8	272.2	179.72	105.88	50.98	24.32	15.835	9.385	4.838	3.361	2.611	2.150
1.30	391.5	195.98	130.71	78.42	39.19	19.663	13.233	8.158	4.379	3.088	2.417	1.9980
1.50	264.7	134.04	90.33	55.20	28.63	15.125	10.533	6.771	3.802	2.728	2.154	1.7898
2.00	172.43	88.40	60.23	37.50	20.17	11.186	8.041	5.374	3.149	2.299	1.8305	1.5286
2.50	143.96	74.22	50.80	31.88	17.394	9.824	7.143	4.840	2.878	2.113	1.6879	1.4120
3.00	130.87	67.68	46.44	29.27	16.087	9.168	6.703	4.572	2.737	2.015	1.6120	1.3496
3.50	123.64	64.06	44.03	27.82	15.356	8.797	6.452	4.417	2.654	1.9573	1.5668	1.3123

Table 3. The function s_1/r of formula (15).

w_2 \ w_1	1.01	1.02	1.03	1.05	1.10	1.20	1.30	1.50	2.00	2.50	3.00	3.50
1.01	6744	2554	1378.0	606.7	192.07	64.19	36.96	21.56	14.047	12.372	11.704	11.363
1.02	4064	1708.5	986.8	472.3	164.10	58.00	33.82	19.669	12.564	10.949	10.302	9.970
1.03	2917	1287.3	770.7	387.6	143.62	53.20	31.48	18.403	11.662	10.104	9.475	9.152
1.05	1875.2	866.8	538.7	286.7	115.51	45.94	27.94	16.589	10.476	9.021	8.427	8.119
1.10	1006.0	484.7	312.3	176.39	78.71	34.80	22.24	13.718	8.759	7.509	6.987	6.714
1.20	537.8	265.4	174.84	102.74	49.51	24.10	16.250	10.547	6.937	5.962	5.542	5.318
1.30	375.1	186.79	124.11	74.09	36.89	18.780	13.027	8.714	5.863	5.062	4.711	4.522
1.50	240.9	120.91	80.94	49.00	25.14	13.365	9.540	6.597	4.570	3.977	3.711	3.567
2.00	134.69	68.03	45.61	28.05	14.750	8.137	5.960	4.254	3.044	2.678	2.512	2.421
2.50	96.41	48.79	32.92	20.23	10.724	5.989	4.426	3.196	2.317	2.050	1.9273	1.8599
3.00	75.99	38.50	26.00	16.003	8.514	4.782	3.549	2.578	1.8817	1.6690	1.5717	1.5181
3.50	63.08	31.97	21.60	13.309	7.095	3.998	2.974	2.167	1.5883	1.4113	1.3302	1.2855

Table 4. The function s_2/r of formula (15).

w_2 \ w_1	1.01	1.02	1.03	1.05	1.10	1.20	1.30	1.50	2.00	2.50	3.00	3.50
1.01	0.0003030	0.0008102	0.0015216	0.003558	0.012245	0.04524	0.09966	0.2767	1.1741	2.861	5.521	9.346
1.02	0.0005081	0.0012245	0.002149	0.004626	0.014490	0.05020	0.10786	0.2931	1.2239	2.967	5.713	9.657
1.03	0.0007152	0.0016430	0.002784	0.005705	0.016761	0.05522	0.11616	0.3098	1.2745	3.075	5.907	9.972
1.05	0.0011356	0.002493	0.004071	0.007896	0.02137	0.06543	0.13306	0.3438	1.3777	3.295	6.303	10.617
1.10	0.002222	0.004689	0.007401	0.013564	0.03333	0.09194	0.17704	0.4326	1.6481	3.873	7.347	12.316
1.20	0.004548	0.009394	0.014538	0.02573	0.05904	0.14922	0.2724	0.6260	2.242	5.147	9.652	16.072
1.30	0.007078	0.014514	0.02231	0.03899	0.08715	0.2122	0.3775	0.8405	2.906	6.577	12.245	20.31
1.50	0.012750	0.02600	0.03976	0.06881	0.15054	0.3548	0.6170	1.3321	4.442	9.899	18.282	30.17
2.00	0.03049	0.06197	0.09445	0.16244	0.3505	0.8081	1.3821	2.917	9.450	20.79	38.13	62.68
2.50	0.05332	0.10828	0.16491	0.2832	0.6091	1.3971	2.380	4.994	16.061	35.21	64.47	105.85
3.00	0.08122	0.16491	0.2511	0.4310	0.9260	2.120	3.606	7.552	24.23	53.05	97.07	159.31
3.50	0.11421	0.2319	0.3530	0.6058	1.3009	2.976	5.059	10.586	33.92	74.24	135.81	222.9

Table 5. The function r/e of formula (16).

The scaling factor e is found from Table 5 or Fig. 4, representing

$$\frac{r}{e} = \frac{(w_1^2\, w_2^4 - 1)\,(w_1^2 - 1)}{2\, w_1\, (w_2^2 + 1)}. \tag{16}$$

The expressions (15, 16) are exact. If necessary, the errors of reading the graphs or of interpolation from the tables may be eliminated by the usual approximation methods starting with the approximate solution.

4. Numerical examples

As numerical examples of the above theory, we take the three slit systems of Fig. 5, representing typical cases of the problems of each of the paragraphs 1, 2, and 3.

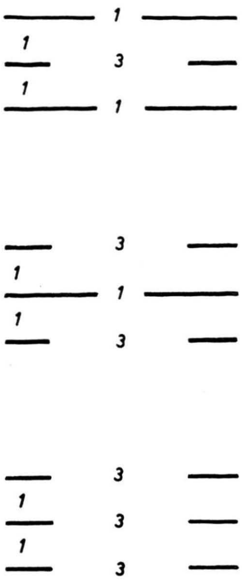

Fig. 5 a, b, c. Numerical examples.

The values of w_1, w_2, and e are found in first and second approximation from the appropriate formulae or tables in the respective paragraphs 1, 2, and 3. The results are given in the Tables 6 a, b, c.

These values of w_1, w_2 and e, substituted in the formulae (5) provide the following data for the slit widths and electrode distances (s. Table 7 a, b, c).

Fig. 5a	First approximation (Formulae 7)	Second approximation (Formulae 8)
w_1	4.2878	4.4556
w_2	1.2093	1.2443
e	0.009298	0.0082386

Table 6 a.

Fig. 5b	First approximation (Formulae 12)	Second approximation (Formulae 13)
w_1	1.3995	1.4251
w_2	5.2029	5.1392
e	0.017530	0.017023

Table 6 b.

Fig. 5c	Fig. 3 and 4, and subsequent approximation
w_1	1.7048
w_2	1.6319
e	0.106336

Table 6 c.

Fig. 5a	Nominal value	First approximation	Second approximation
$2s_1$	1.0000	0.9861	1.0000
$2s_2$	3.0000	2.9580	3.0095
πr	1.0000	0.9215	1.0011

Table 7 a.

Fig. 5b	Nominal value	First approximation	Second approximation
$2s_1$	3.0000	3.0262	3.0015
$2s_2$	1.0000	0.9882	1.0001
πr	1.0000	0.9637	0.9990

Table 7 b.

Fig. 5c	Nominal value	Table 6c
$2s_1$	3.00000	3.00001
$2s_2$	3.00000	2.99994
πr	1.00000	1.00000

Table 7 c.

The best values of w_1, w_2, and e are used to compute $b = w_1 w_2^2 e$ and $d = w_1 e$. For the three different cases we get the following results (s. Table 8).

	Fig. 5a	Fig. 5b	Fig. 5c
b	0.056834	0.64073	0.48277
d	0.036708	0.024259	0.18128

Table 8.

With these values of b and d, the potential distributions and field strengths were calculated along the main axis of the slit systems of Fig. 5 a, b, and c for the following potentials on the electrodes:

(1) all electrodes on zero potential, but an external field of one unit of field strength right hand side of the slit system;

(2) first electrode on potential $+1$, second and third electrodes zero;

(3) second electrode on potential $+1$, first and third electrodes zero;

(4) first half electrode on potential $+1$, other half of the first electrode and second and third electrodes zero;

(5) second half electrode on potential $+1$, other half of the second electrode and first and third electrodes zero.

In the cases (4) and (5) only the crossfield was computed.

y	$V(y)$ (1)	$V(y)$ (2)	$V(y)$ (3)	$\frac{\partial V}{\partial y}$ (1)	$\frac{\partial V}{\partial y}$ (2)	$\frac{\partial V}{\partial y}$ (3)	$\frac{\partial V}{\partial x}$ (4)	$\frac{\partial V}{\partial x}$ (5)
− 20.86	0.00010	0.00112	0.00061	0.00000	0.00005	0.00003	0.00000	0.00000
− 10.43	0.00020	0.00224	0.00123	0.00002	0.00021	0.00012	0.00000	0.00000
− 4.158	0.00050	0.00560	0.00307	0.00012	0.00134	0.00073	0.00001	0.00001
− 2.057	0.00100	0.01120	0.00614	0.00047	0.00529	0.00290	0.00005	0.00007
− 0.984	0.00200	0.02240	0.01225	0.00181	0.02030	0.01108	0.00036	0.00050
− 0.270	0.00500	0.05587	0.03032	0.00888	0.09872	0.05250	0.00434	0.00596
+ 0.082	0.01000	0.11085	0.05847	0.02046	0.2223	0.10803	0.01956	0.02544
0.449	0.02000	0.2154	0.10218	0.03383	0.3372	0.11523	0.05934	0.06392
1.064	0.05000	0.4593	0.13751	0.07031	0.4440	− 0.01691	0.19529	0.09556
1.526	0.10000	0.6710	0.10505	0.16659	0.4556	− 0.11251	0.4008	0.06550
1.898	0.20000	0.82374	0.06068	0.40745	0.3410	− 0.11073	0.6000	0.02732
2.377	0.5000	0.92788	0.02546	0.80373	0.11484	− 0.04011	0.5053	0.00378
2.938	1.0000	0.96386	0.01278	0.94207	0.03398	− 0.01199	0.2989	0.00056
3.969	2.000	0.98192	0.00639	0.98483	0.00890	− 0.00315	0.15662	0.00007
6.988	5.000	0.99276	0.00256	0.99754	0.00144	− 0.00051	0.06350	0.00001
11.99	10.000	0.99638	0.00128	0.99938	0.00036	− 0.00013	0.03181	0.00000
22.00	20.00	0.99819	0.00063	0.99985	0.00009	− 0.00003	0.01591	0.00000
52.00	50.00	0.99927	0.00024	0.99998	0.00001	− 0.00001	0.00637	0.00000

Table 9 a.

y	$V(y)$ (1)	$V(y)$ (2)	$V(y)$ (3)	$\frac{\partial V}{\partial y}$ (1)	$\frac{\partial V}{\partial y}$ (2)	$\frac{\partial V}{\partial y}$ (3)	$\frac{\partial V}{\partial x}$ (4)	$\frac{\partial V}{\partial x}$ (5)
− 155.0	0.00010	0.00010	0.00252	0.00000	0.00000	0.00001	0.00000	0.00000
− 77.71	0.00020	0.00020	0.00504	0.00000	0.00000	0.00003	0.00000	0.00000
− 31.08	0.00050	0.00050	0.01262	0.00002	0.00001	0.00020	0.00000	0.00000
− 15.52	0.00100	0.00099	0.02524	0.00006	0.00003	0.00081	0.00000	0.00003
− 7.715	0.00200	0.00199	0.05037	0.00026	0.00013	0.00320	0.00000	0.00027
− 2.970	0.00500	0.00497	0.12442	0.00154	0.00077	0.01894	0.00001	0.00406
− 1.286	0.01000	0.00994	0.2390	0.00556	0.00276	0.05960	0.00004	0.02566
− 0.298	0.02000	0.01988	0.4190	0.01771	0.00879	0.12954	0.00027	0.11377
+ 0.501	0.05000	0.04968	0.6627	0.07573	0.03739	0.15194	0.00292	0.3873
0.892	0.10000	0.09856	0.7499	0.2007	0.09733	0.04903	0.01519	0.5881
1.238	0.2000	0.19262	0.7305	0.3739	0.16926	− 0.09813	0.05283	0.5336
1.860	0.5000	0.4219	0.5473	0.5743	0.17732	− 0.15962	0.13837	0.2264
2.606	1.0000	0.6372	0.3473	0.7575	0.10953	− 0.10368	0.17094	0.07003
3.787	2.000	0.80263	0.18965	0.90890	0.04203	− 0.04027	0.13119	0.01344
6.913	5.000	0.91887	0.07804	0.98300	0.00789	− 0.00759	0.06157	0.00101
11.96	10.000	0.95927	0.03918	0.99565	0.00202	− 0.00195	0.03156	0.00013
21.98	20.00	0.97961	0.01961	0.99890	0.00051	− 0.00049	0.01588	0.00002
51.99	50.00	0.99184	0.00783	0.99982	0.00008	− 0.00008	0.00637	0.00000

Table 9 b.

y	$V(y)$ (1)	$V(y)$ (2)	$V(y)$ (3)	$\frac{\partial V}{\partial y}$ (1)	$\frac{\partial V}{\partial y}$ (2)	$\frac{\partial V}{\partial y}$ (3)	$\frac{\partial V}{\partial x}$ (4)	$\frac{\partial V}{\partial x}$ (5)
− 875	0.00010	0.00013	0.00022	0.00000	0.00000	0.00000	0.00000	0.00000
− 437.0	0.00020	0.00026	0.00044	0.00000	0.00000	0.00000	0.00000	0.00000
− 175.0	0.00050	0.00066	0.00110	0.00000	0.00000	0.00000	0.00000	0.00000
− 87.51	0.00100	0.00132	0.00219	0.00001	0.00001	0.00001	0.00000	0.00000
− 43.75	0.00200	0.00264	0.00438	0.00005	0.00003	0.00005	0.00000	0.00000
− 17.47	0.00500	0.00659	0.01096	0.00029	0.00019	0.00031	0.00000	0.00001
− 8.694	0.01000	0.01318	0.02190	0.00114	0.00075	0.00124	0.00002	0.00009
− 4.260	0.02000	0.02636	0.04360	0.00445	0.00293	0.00479	0.00012	0.00073
− 1.463	0.05000	0.06570	0.10563	0.02465	0.01608	0.02414	0.00167	0.00943
− 0.324	0.10000	0.13003	0.19088	0.07295	0.04612	0.05209	0.00955	0.04462
+ 0.544	0.2000	0.2500	0.2812	0.16957	0.09543	0.03886	0.03953	0.10862
1.615	0.5000	0.5112	0.2674	0.4177	0.13286	− 0.04766	0.13760	0.09739
2.512	1.0000	0.7137	0.17216	0.6904	0.08604	− 0.04747	0.17821	0.03454
3.748	2.000	0.84921	0.09324	0.89039	0.03232	− 0.01958	0.13391	0.00665
6.898	5.000	0.93872	0.03821	0.98011	0.00597	− 0.00371	0.06182	0.00049
11.95	10.000	0.96929	0.01917	0.99493	0.00153	− 0.00095	0.03160	0.00006
21.97	20.00	0.98463	0.00960	0.99873	0.00038	− 0.00024	0.01589	0.00000
51.99	50.00	0.99385	0.00384	0.99980	0.00006	− 0.00004	0.00636	0.00000

Table 9 c.

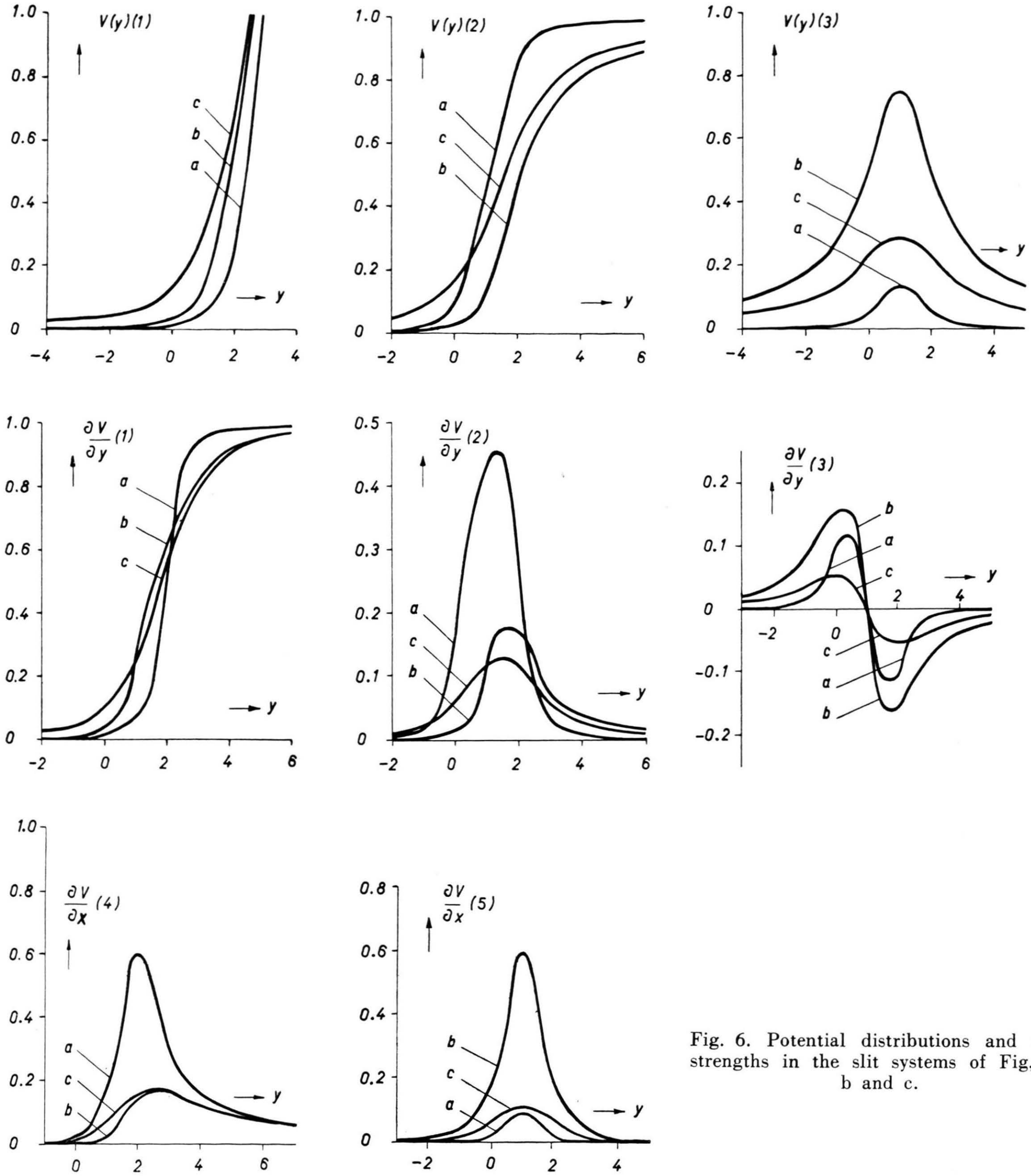

Fig. 6. Potential distributions and field strengths in the slit systems of Fig, 5 a, b and c.

Because of the symmetry of the slit systems with regard to the second electrode, all other cases of potentials set on the electrodes and external fields, can be found as linear superpositions of the above cases.

For reason of completeness we give a list of the formulae, used to compute the potentials and field strengths. These formulae are only valid along the main axis.

$$y = v + \frac{2}{\pi} \arctan \frac{v}{b} + \frac{2}{\pi} \arctan \frac{v}{d} - \frac{bd}{v} ;$$

$$V(y)\,(1) = v\,, \quad V(y)\,(2) = \frac{2}{\pi} \arctan \frac{v}{b}\,,$$

$$V(y)\,(3) = \frac{2}{\pi} \left\{ \arctan \frac{v}{d} - \arctan \frac{v}{b} \right\} ;$$

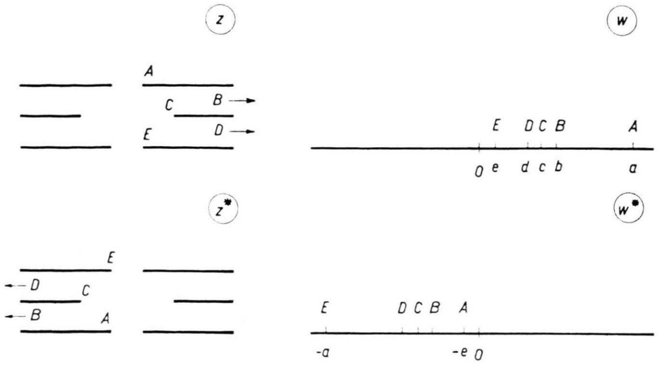

Fig. 7. The slit system in the various planes.

$$\frac{\partial y}{\partial v} = 1 + \frac{2}{\pi}\frac{b}{v^2+b^2} + \frac{2}{\pi}\frac{d}{v^2+d^2} + \frac{b\,d}{v^2} = \frac{\partial x}{\partial u},$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial v} \Big/ \frac{\partial y}{\partial v}, \quad \frac{\partial V}{\partial x} = \frac{\partial V}{\partial u} \Big/ \frac{\partial x}{\partial u},$$

$$\frac{\partial V}{\partial v}(1) = 1, \quad \frac{\partial V}{\partial v}(2) = \frac{2}{\pi}\frac{b}{v^2+b^2},$$

$$\frac{\partial V}{\partial v}(3) = \frac{2}{\pi}\left\{\frac{d}{v^2+d^2} - \frac{b}{v^2+b^2}\right\},$$

$$\frac{\partial V}{\partial u}(4) = \frac{1}{\pi}\frac{v}{v^2+b^2},$$

$$\frac{\partial V}{\partial u}(5) = \frac{1}{\pi}\left\{\frac{v}{v^2+d^2} - \frac{v}{v^2+b^2}\right\}.$$

The results for the three slit systems are given in Table 9 a, b, and c, and plotted in Fig. 6.

Appendix

Proof, that $a/b = d/e$ and $b/c = c/d$ if the central electrode is a plane of symmetry.

The cross section of the slit system with a z-plane perpendicular to the direction of the slits is considered as a degenerate polygon to be mapped conformally by the SCHWARZ–CHRISTOFFEL transformation on the real axis of a w-plane.

Let the inverse of the mapping function be

$$z = z(w).$$

Again the w-plane is mapped conformally according to

$$w^* = -a\,e/w.$$

Through this transformation the real axis remains invariant, the points a, b, c, d and e being mapped on the respective points

$$-\frac{a\,e}{a} = -e, \quad -\frac{a\,e}{b}, \quad -\frac{a\,e}{c}, \quad -\frac{a\,e}{d}, \quad -\frac{a\,e}{e} = -a.$$

Now the projection of the slit system is rotated over 180°. This corresponds to the transformation

$$z^* = -z.$$

If, however, the slit system has the supposed symmetry, it remains invariant through this transformation. The rotated slit system is mapped again on a w-plane, producing an image identical to the one in the w^*-plane. This implies, that

$$-\frac{a\,e}{b} = -d, \quad -\frac{a\,e}{c} = -c,$$

or $$a/b = d/e = w_1, \qquad b/c = c/d = w_2.$$

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